Embeddings and chains of free groups

Eric Jaligot, Azadeh Neman

May 27, 2008

Abstract

We build two nonabelian CSA-groups in which maximal abelian subgroups are conjugate and divisible, as the countable unions of increasing chains of CSA-groups and by keeping the constructions as free as possible in each case.

For $n \geq 1$, let F_n denote the free group on n generators. We view all groups G as first-order structures $\langle G, \cdot, {}^{-1}, 1 \rangle$, where $\cdot$, ${}^{-1}$, and 1 denote respectively the multiplication, the inverse, and the identity of the group. The following striking results are proved in a series of papers of Sela culminating in [Sel07].

Fact 1 [Sel05, Sel06a, Sel06b, Sel07]

(1) *For any $2 \leq n \leq m$, the natural embedding $F_n \leq F_m$ is an elementary embedding.*

(2) *For any $n \geq 2$, the (common) complete theory $\mathrm{Th}(F_n)$ is stable.*

We refer to [Hod93] for model theory in general, and to [Poi87] and [Wag97] for stability theory and in particular stable groups.

Let F denote the free group over countably many generators. Fact 1 has the following corollary.

Corollary 2 *The natural embeddings $F_2 \leq \cdots F_n \leq \cdots \leq F$ are all elementary. In particular each F_n is an elementary substructure of F, and $\mathrm{Th}(F)$ is stable.*

A *CSA-group* is a group in which maximal abelian subgroups A are *malnormal*, i.e., such that $A \cap A^g \neq 1$ implies that g is in A for any element g of the ambient group. The class of CSA-groups contains all free groups and is studied from various points of view. We refer to [JOH04, JMN08] for a model theoretic approach in combination of questions concerning particular groups [Che79, Jal01, Cor03], and to [KMRS08] for an approach more related to computational aspects in limit groups.

We prove the following lemma on embeddings of torsion-free CSA-groups in which maximal abelian subgroups are cyclic.

Lemma 3 *Let $k \geq 2$, and let G be a countable torsion-free CSA-group in which maximal abelian subgroups are cyclic. Let a_0 be any generator of a maximal abelian subgroup of G. Then G embeds into a torsion-free CSA-group $H = \langle F, r \rangle$ in which maximal abelian subgroups are cyclic, where F is a free-group over countably many generators and $r^k = a_0$, and where maximal abelian subgroups of G are H-conjugate. In particular any element of G has a k-th root in H.*

PROOF:

G has countably many maximal abelian subgroups, and countably many conjugacy classes of such maximal abelian subgroups, which can be enumerated by $i < \omega$.

For each such conjugacy class, fix a maximal (cyclic) abelian subgroup A_i, and inside A_i fix a generator a_i. Notice that we can always take for a_0 any given generator of a maximal abelian subgroup of G.

We define inductively on i an increasing family of supergroups G_i of G as follows.

- $G_0 = G$.
- G_{i+1} is the HNN-extension $\langle G_i, t_i \mid a_i^{t_i} = a_{i+1} \rangle$.

We note that each G_{i+1} is generated by G_0 together with the elements t_0, ..., t_i.

Let now

$$G_\omega = \bigcup_{i<\omega} G_i.$$

It is clear that G_ω is generated by G_0, together with the elements t_i's. By construction, it is also clear that in G_ω any two distinct elements t_i and t_j satisfy no relation by [LS77, Britton's Lemma]. In particular they generate a free group on two generators t_i and t_j.

We also see that in G_ω any two maximal abelian subgroups of G_0 are conjugate, actually by the subgroup of G_ω generated by all the elements t_i. In particular in G_ω one has

$$G_0 \subseteq \langle a_0 \rangle^{\langle (t_i)_{i<\omega} \mid \rangle}$$

and G_ω is generated by $\langle a_0 \rangle$ and $\langle (t_i)_{i<\omega} \mid \rangle$.

Consider now an abelian torsion-free cyclic supergroup R of $A_0 = \langle a_0 \rangle$ generated by an element r such that $r^k = a_0$. As $k \geq 2$ by assumption, r does not belong to the cyclic subgroup $\langle a_0 \rangle$ of R.

Now one can form the free product of R and G_ω with amalgamated subgroup $\langle a_0 \rangle$, say

$$G_{\omega+1} = R *_{\langle a_0 \rangle} G_\omega.$$

As $r^k = a_0$ and G_ω is generated by a_0 and the t_i's, one gets that $G_{\omega+1}$ is generated by the t_i's together with r. Hence

$$G_{\omega+1} = \langle r, (t_i)_{i<\omega} \rangle$$

and the second set of generators freely generate the free group F.

Now the natural embedding

$$G \simeq G_0 \leq G_{\omega+1} \simeq H$$

is the desired embedding.

We note that the class of CSA-groups is inductive, as pointed out in [JOH04]. This follows indeed from the fact that the class is axiomatizable by universal axioms. As H is a direct limit of CSA-groups, it is also a CSA-group. We note that maximal abelian subgroups of H are also cyclic by the results of [JOH04]. $\square$

By Lemma 3, one can find an infinite sequence of embeddings

$$^1G \;\leq\; ^2G \;\leq \cdots \leq\; ^{k-1}G \;\leq\; ^kG \;\leq \cdots$$

where ^1G is a nonabelian countable free group and such that for each $k \geq 2$ the embedding $^{k-1}G \;\leq\; ^kG$ is as in Lemma 3, i.e., such that maximal abelian subgroups of ^{k-1}G are conjugate in ^kG and each element in ^{k-1}G is k-divsible in ^kG. We note that each group ^kG is a torsion-free CSA-group in which maximal abelian subgroups are cyclic.

Consider now the group

$$G = \bigcup_{k \geq 1} {}^kG$$

We see that G is a CSA-group, as the class of CSA-groups is inductive. As maximal abelian subgroups coincide with centralizers of nontrivial elements in CSA-groups, one sees that maximal abelian subgroups of G are conjugate by construction. Again the construction implies that each element g of G is n-divisible for each n, and as maximal abelian subgroups coincide with centralizers of nontrivial elements one concludes that maximal abelian subgroups are divisible.

Of course G is nonabelian since it contains the nonabelian free group ^1G, and we note that it is also torsion-free. We obtain thus a non-abelian CSA-group in which maximal abelian subgroups are conjugate and divisible, as a countable union of torsion-free CSA-groups in which maximal abelian subgroups are cyclic.

As the free group is stable, one may wonder about the stability of the group G built above. Recall that the *stable* sets are the sets defined by a formula $\varphi(\overline{x}, \overline{y})$ for which there exists an integer n, called the *ladder index* of φ, bounding uniformly the size of n*-ladders* of φ, that is the sets of tuples

$$(\overline{x_1}, \cdots, \overline{x_1} \;;\; \overline{y_1}, \cdots, \overline{y_n})$$

such that $\varphi(\overline{x_i}, \overline{x_j})$ is true if and only if $i \leq j$. We refer to [Hod93]. We recall also that by Ramsey's theorem boolean combinations of stable sets are stable [Wag97, 0.2.10], and that the replacement of variables by parameters obviously does not affect the stability of definable sets.

To check the stability of quantifier-free definable sets, the following lemma may be relevant for the type of groups considered here.

Fact 4 *Let $\varphi(\overline{x}, \overline{y})$ be a quantifier-free formula in the langage of groups, that is a boolean combination of equations into the variables involved in the tuples $\overline{x}$ and $\overline{y}$. If a group G is a union of an increasing family of subgroups G_i, and if φ defines a stable set in each G_i with a uniform bound on the ladder indices of φ in each G_i, then φ defines a stable set in G.*

PROOF:

By assumption, there exists a uniform bound on the ladder index of φ in G_i, when i varies, and thus we find an integer n such that no subgroup G_i can have a ladder of size n.

Now one sees that G cannot have a ladder of size n also, as otherwise all elements of the tuples involved in such a ladder would belong to one of the subgroups G_i, a contradiction as φ is quantifier-free.

This proves that φ defines a stable set in G. $\square$

Another construction of a non-abelian CSA-group in which maximal abelian subgroups are conjugate and divisible and with a temptative to keep certain parts of the stability of the free group is as follows. It now consists of adding all roots simultaneously.

Instead of starting from the free group F, let us start with ${}^1G = \mathbb{Q} * F$. Fix a_0 the element corresponding to the element 1 of $\mathbb{Q}$ (in additive notation). Passing from ${}^1G$ to ${}^2G$ is now done as follows. Enumerate by a_1, a_2, ... etc, generators of maximal abelian cyclic subgroups of ${}^1G$, picking up exactly one maximal abelian cyclic subgroup in each conjugacy class of such subgroups. Now embed $\langle a_0 \rangle$ in a copy of $\mathbb{Q}$, in such a way that a_0 represents 1 in $\mathbb{Q}$ (in additive notation), and form the free product of ${}^1G$ and this new copy of $\mathbb{Q}$, with amalgamated subgroup $\langle a_0 \rangle$. One gets then a new CSA-group, and one can conjugate a_0 to a_1 by forming an appropriate HNN-extension, with an element t_0. One repeats this process as in Lemma 3, obtaining CSA-groups at each step by the results of [JOH04]. Calling ${}^2G$ the union, one gets a CSA-group with one conjugacy class of divisible maximal abelian subgroups, and generated by one such subgroup and a countable free group (generated by all the t_i's added when forming the successive HNN-extensions).

One can then build similarly an infinite sequence of embeddings

$$ {}^1G \;\leq\; {}^2G \;\leq \cdots \leq\; {}^{k-1}G \;\leq\; {}^kG \;\leq \cdots $$

such that for each $k \geq 2$ the embedding ${}^{k-1}G \;\leq\; {}^kG$ is as in the process described above. In particular each ${}^kG$ is generated by a divisible abelian group isomorphic to $\mathbb{Q}$ and a free group F. Again the union of all these groups is a CSA-group in which maximal abelian subgroups are conjugate and divisible.

We thank Vincent Guirardel, Abderezak Ould Houcine, Zlil Sela, and Alina Vdovina who pointed out a mistake in the first version of the present work.

References

[Che79] G. Cherlin. Groups of small Morley rank. *Ann. Math. Logic*, 17(1-2):1–28, 1979.

[Cor03] L. J. Corredor. $(*)$-groups and pseudo-bad groups. *Rev. Colombiana Mat.*, 37(2):51–63, 2003.

[Hod93] W. Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[Jal01] E. Jaligot. Full Frobenius groups of finite Morley rank and the Feit-Thompson theorem. *Bull. Symbolic Logic*, 7(3):315–328, 2001.

[JMN08] E. Jaligot, A. Muranov, and A. Neman. Independence property and hyperbolic groups. *Bull. Symbolic Logic*, 14(1):88–98, 2008.

[JOH04] E. Jaligot and A. Ould Houcine. Existentially closed CSA-groups. *J. Algebra*, 280(2):772–796, 2004.

[KMRS08] O. Kharlampovich, A. Myasnikov, V. Remeslennikov, and D. Serbin. Exponential extensions of groups. *J. Group Theory*, 11(1):119–140, 2008.

[LS77] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[Poi87] B. Poizat. *Groupes stables*. Bruno Poizat, Lyon, 1987. Une tentative de conciliation entre la géométrie algébrique et la logique mathématique. [An attempt at reconciling algebraic geometry and mathematical logic].

[Sel05] Z. Sela. Diophantine geometry over groups. V_1. Quantifier elimination. I. *Israel J. Math.*, 150:1–197, 2005.

[Sel06a] Z. Sela. Diophantine geometry over groups. V_2. Quantifier elimination. II. *Geom. Funct. Anal.*, 16(3):537–706, 2006.

[Sel06b] Z. Sela. Diophantine geometry over groups. VI. The elementary theory of a free group. *Geom. Funct. Anal.*, 16(3):707–730, 2006.

[Sel07] Z. Sela. Diophantine geometry over groups VIII: Stability. preprint: http://www.ma.huji.ac.il/~zlil/, 2007.

[Wag97] F. O. Wagner. *Stable groups*. Cambridge University Press, Cambridge, 1997.